

A CHARACTERIZATION OF INOUE SURFACES

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ABSTRACT. We give a characterization of Inoue surfaces in terms of automorphic pluriharmonic functions on a cyclic covering. Together with results of Chiose and Toma, this completes the classification of compact complex surfaces of Kähler rank one.

In this paper we shall prove a conjecture proposed in [C-T]:

Theorem 0.1. *Let S be a compact connected complex surface of algebraic dimension 0. Suppose that there exists an infinite cyclic covering $\tilde{S} \xrightarrow{\pi} S$ (with covering transformations generated by $\varphi \in \text{Aut}(\tilde{S})$) and a nonconstant positive pluriharmonic function F on \tilde{S} such that*

$$F \circ \varphi = \lambda \cdot F$$

for some $\lambda \in \mathbb{R}^+$. Then S is a (possibly blown up) Inoue surface.

The class of Inoue surfaces was discovered by Inoue (and independently Bombieri) around 1972 [Ino] [Nak]. They are special (and explicit) compact quotients of $\mathbb{H} \times \mathbb{C}$, and they enjoy the following properties:

- the first Betti number is 1, the second Betti number is 0;
- they admit holomorphic foliations;
- they do not contain compact complex curves.

Conversely, Inoue proved in [Ino] that any compact connected complex surface with the above properties is a Inoue surface.

Our proof of Theorem 0.1 will be ultimately a reduction to Inoue's theorem. The pluriharmonic function F naturally induces a holomorphic (and possibly singular) foliation \mathcal{F} on S . By a “topological” study of such a foliation we will be able to understand some topological structure of S , and in particular to show that $c_2(S_{\min}) = 0$ or $c_1^2(S_{\min}) = 0$ (where S_{\min} denotes the minimal model of S). From this vanishing of Chern numbers, and results of Kodaira and Inoue, the conclusion will be immediate. Remark that, conversely and by construction, every Inoue surface satisfies the hypotheses of Theorem 0.1, which therefore gives a precise characterization of Inoue surfaces.

Together with the results of [C-T], Theorem 0.1 allows to complete the classification of compact surfaces of Kähler rank one. Recall [H-L] [C-T] that a compact connected complex surface S has *Kähler rank one* if it is not Kählerian but it admits a closed semipositive $(1, 1)$ -form, not identically

vanishing (this is not the original definition of [H-L], but it is equivalent to it by the results of [C-T], see also [Lam] and [Tom]).

Corollary 0.2. ([C-T] + Theorem 0.1). *The only compact connected complex surfaces of Kähler rank one are:*

- (1) *Non-Kählerian elliptic fibrations;*
- (2) *Certain Hopf surfaces, and their blow-ups;*
- (3) *Inoue surfaces, and their blow-ups.*

In the case of a Inoue surface, a closed semipositive $(1,1)$ -form is given by $(dF/F) \wedge (d^c F/F)$, with F as in Theorem 0.1.

1. GEOMETRIC PRELIMINARIES

Let S be a surface as in Theorem 0.1. Without loss of generality, we may assume that S is minimal, since the hypotheses are clearly bimeromorphically invariant. The assumption $a(S) = 0$ implies that S belongs to the class VII_o [BPV, p. 188], that is $b_1(S) = 1$ and $kod(S) = -\infty$: the existence of a positive nonconstant pluriharmonic function on some covering of S excludes the case of tori and K3 surfaces. For the same reason, S cannot be a Hopf surface.

We claim that, in order to prove Theorem 0.1, it is sufficient to prove that

$$c_2(S) = 0$$

or

$$c_1^2(S) = 0.$$

Indeed, we firstly observe that these two conditions are equivalent, by Noether formula and $\chi(\mathcal{O}_S) = 0$ (which follows from $S \in \text{VII}_o$). Then, $c_2(S) = 0$ and $b_1(S) = 1$ imply $b_2(S) = 0$. By a classical result of Kodaira [Nak, Th. 2.4], S contains no compact complex curve, otherwise it would be a Hopf surface. Since S also admits a holomorphic foliation (see below), all the hypotheses of Inoue's theorem [Ino] are satisfied and we get that S is a Inoue surface.

The automorphic function F on \tilde{S} induces a real analytic map

$$f = \log F : S \longrightarrow \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \cdot \log \lambda.$$

The regular fibers of f are smooth Levi-flat hypersurfaces in S , because F is pluriharmonic. However, f could have also some singular fibers, corresponding to critical points of F . In fact, our aim is precisely to show that these singular fibers do not exist at all, since this is clearly equivalent to $c_2(S) = 0$. In order to avoid some cumbersome statement, we shall suppose that the fibers of f are connected. The general case requires only few straightforward modifications of the proof below.

The holomorphic 1-form $\omega = \partial F \in \Omega^1(\tilde{S})$ descends to S to a holomorphic section (still denoted by ω) of $\Omega^1(S) \otimes L$, where L is a flat line bundle (the one defined by the cocycle $\lambda \in \mathbb{R}^+ \subset \mathbb{C}^* = H^1(S, \mathbb{C}^*)$). This twisted closed holomorphic 1-form induces a holomorphic foliation \mathcal{F} on S , which is tangent to the fibers of f .

In the following it will be important to distinguish between the singularities of \mathcal{F} , $\text{Sing}(\mathcal{F})$, and the zeroes of ω , $Z(\omega)$. The former are only isolated points, since (as customary) we like to deal with “saturated” foliations. The latter, on the contrary, may contain some compact complex curves. Remark also that $Z(\omega)$ coincides with the set of critical points of f , $\text{Crit}(f)$.

The foliation \mathcal{F} has a normal bundle $N_{\mathcal{F}}$ and a tangent bundle $T_{\mathcal{F}}$ [Br1], which are related to the canonical bundle K_S of S by the adjunction type relation

$$N_{\mathcal{F}} \otimes T_{\mathcal{F}} = K_S^{-1}.$$

Because \mathcal{F} is generated by $\omega \in \Omega^1(S) \otimes L$, we have

$$N_{\mathcal{F}} = L \otimes \mathcal{O}(-\sum m_j C_j)$$

where $\{C_j\}$ are the curves contained in $Z(\omega)$ (if any) and $\{m_j\}$ are the respective vanishing orders.

We shall prove below that $Z(\omega)$ is at most composed by isolated points, giving by the previous formula the flatness of $N_{\mathcal{F}} = L$. Then we shall prove that either $c_2(S) = 0$ or $T_{\mathcal{F}}$ is also flat. But in this second case we therefore get that K_S is flat too, hence $c_1^2(S) = 0$.

2. THE STRUCTURE OF THE REGULAR FIBERS

Here we consider a regular fiber of f ,

$$M_{\vartheta} = \{f = \vartheta\}, \quad \vartheta \text{ regular value,}$$

and prove that it has the expected structure.

Proposition 2.1. *The leaves of $\mathcal{F}|_{M_{\vartheta}}$ are either all isomorphic to \mathbb{C} , or all isomorphic to \mathbb{C}^* . In the first case, M_{ϑ} is diffeomorphic to \mathbb{T}^3 , and $\mathcal{F}|_{M_{\vartheta}}$ is a linear totally irrational foliation. In the second case, M_{ϑ} is a \mathbb{S}^1 -bundle over \mathbb{T}^2 , and $\mathcal{F}|_{M_{\vartheta}}$ is the pull-back of a linear irrational foliation on \mathbb{T}^2 .*

The first case will lead to Inoue surface of type S_M , and the second case to those of type $S_{N,p,q,r;t}^{(+)}$ or $S_{N,p,q,r}^{(-)}$ [Ino].

Proof. The foliation $\mathcal{F}_{\vartheta} = \mathcal{F}|_{M_{\vartheta}}$ is defined by the closed and nonsingular 1-form $\beta = d^c F|_{M_{\vartheta}}$ (which is well defined on a neighbourhood of any fiber, up to a multiplicative constant). We may use some classical results of Tischler [God, I.4] concerning the structure of (real) codimension one foliations defined by closed 1-forms. According to those results, the foliation can be smoothly perturbed to a fiber bundle over the circle with fiber Σ_g , the (real) oriented compact surface of genus $g \geq 1$. Note that, since $a(S) = 0$, the leaves of \mathcal{F}_{ϑ} cannot be all compact, and so they are all dense in M_{ϑ} . Moreover, by using the flow of a smooth vector field v on M_{ϑ} such that $\beta(v) \equiv 1$, and the closedness of β , we see that the leaves are all diffeomorphic to the same abelian covering of Σ_g .

The above flow of v sends leaves to leaves, but of course it does not need to preserve the complex structure of the leaves, that is it does not need

to realize a conformal diffeomorphism between the leaves. However, the compactness of M_ϑ implies, at least, that such a diffeomorphism is quasi-conformal. In particular, all the leaves have the same (conformal) universal covering: either they are all parabolic, or all hyperbolic. For our purposes, it is sufficient to prove that the leaves are parabolic: since they are abelian coverings of Σ_g , this implies that $g = 1$, and the rest of the statement is standard [God, I.4].

We can associate to \mathcal{F}_ϑ a closed positive current $\Phi \in A^{1,1}(S)'$, by integration along the leaves against the transverse measure defined by β [Ghy]: if $\eta \in A^{1,1}(S)$, we define

$$\Phi(\eta) = \int_{M_\vartheta} \beta \wedge \eta.$$

Obviously this current does not charge compact complex curves, hence by results of Lamari [Lam] [Tom, Rem. 8] it is an *exact* positive current. As a consequence of this, its De Rham cohomology class $[\Phi]$ has vanishing product with the Chern class of $T_{\mathcal{F}}$:

$$c_1(T_{\mathcal{F}}) \cdot [\Phi] = 0.$$

Let us show that this implies the parabolicity of the leaves (this is a particularly simple instance of the foliated Gauss-Bonnet theorem, see [Ghy]). In the opposite case, we may put on the leaves of \mathcal{F}_ϑ their Poincaré metric, which can be seen as a hermitian metric on $T_{\mathcal{F}}|_{M_\vartheta}$. It is a continuous metric [Ghy], and it can be regularized by a smooth hermitian metric on $T_{\mathcal{F}}|_{M_\vartheta}$ whose curvature along the leaves is still strictly negative (for instance, with the help of the above vector field v). We then extend this hermitian metric to the full $T_{\mathcal{F}}$, on the full S , in any way. The curvature form $\Theta \in A^{1,1}(S)$ clearly satisfies

$$\Phi(\Theta) = \int_{M_\vartheta} \beta \wedge \Theta < 0.$$

This is in contradiction with the vanishing of $c_1(T_{\mathcal{F}}) \cdot [\Phi]$. \square

Remark 2.2. Let us stress a subtle detail of the previous proof. The current Φ can be also considered as a current on the real threefold M_ϑ . As such, however, it is *not* exact. Thus, in order to get the vanishing of $c_1(T_{\mathcal{F}}) \cdot [\Phi]$, we used also the fact that the tangent bundle $T_{\mathcal{F}_\vartheta}$ extends to the full S , or more precisely that its Chern class in $H^2(M_\vartheta, \mathbb{R})$ extends to S , which is obvious in our case since we have a global foliation on S . Now, one can imagine a more general situation, in which we have a Levi-flat hypersurface M in a class VII₀ surface, such that the Levi foliation is given by a closed 1-form (or, more generally, admits a transverse measure invariant by holonomy). Is it still true that the leaves of this Levi foliation are parabolic?

3. THE STRUCTURE OF THE SINGULARITIES

In order to study $\text{Sing}(\mathcal{F})$ and $Z(\omega)$, we need a general lemma on critical points.

Let \tilde{U} be a smooth complex surface and let $D \subset \tilde{U}$ be a compact connected curve (with possibly several irreducible components). Suppose that the intersection form on D is negative definite, so that D is contractible to one point [BPV, p. 72]. After contraction, we get a normal surface U and a point $q \in U$, image of D ; we do not exclude that q is a smooth point. Let now \tilde{H} be a holomorphic function on \tilde{U} , vanishing on D , such that

$$\text{Crit}(\tilde{H}) = D.$$

After contraction, we thus get a holomorphic function H on U with (at most) an isolated critical point at q . If B is a small ball centered at q , then $H_0 = \{H = 0\} \cap B$ is a collection of k discs H_0^1, \dots, H_0^k passing through q , whereas $H_\varepsilon = \{H = \varepsilon\} \cap B$ (ε small and not zero) is a connected curve with k boundary components. The topological type of H_ε does not depend on ε (small and not zero), it is the so-called Milnor fiber of H at q .

Lemma 3.1. *Under the previous notation, suppose that the genus of the Milnor fiber of H at q is zero. Then:*

- (1) q is a smooth point of U ;
- (2) either q is a regular point for H , or it is a critical point of Morse type.

Proof. The hypothesis means that the Milnor fiber is a sphere with k holes. By a standard construction, we may glue to $W = \cup_{|\varepsilon| < r} H_\varepsilon$ ($r > 0$ small) a collection of k bidiscs in such a way that we obtain a normal complex surface V and a proper holomorphic map

$$G : V \longrightarrow \mathbb{D}(r)$$

such that:

- (i) $W \subset V$ and $G|_W = H$;
- (ii) $G_\varepsilon = G^{-1}(\varepsilon)$ is a smooth rational curve for every nonzero $\varepsilon \in \mathbb{D}(r)$;
- (iii) $G_0 = G^{-1}(0)$ is a collection of k rational curves G_0^1, \dots, G_0^k passing through q , with $G_0^j \cap W = H_0^j$ for every j .

Remark that all the components G_0^j of G_0 have multiplicity 1, i.e. G vanishes along $G_0^j \setminus \{q\}$ at first order only. On the other hand, we may blow-up q to the original D , and we get a smooth complex surface \tilde{V} and a map

$$\tilde{G} : \tilde{V} \longrightarrow \mathbb{D}(r)$$

whose fiber over 0 is $\hat{G}_0^1 \cup \dots \cup \hat{G}_0^k \cup D$, with \hat{G}_0^j the strict transform of G_0^j . By construction, we have

$$\text{mult}(\hat{G}_0^j) = 1$$

for every j and

$$\text{mult}(C) \geq 2$$

for every irreducible component C of D , since $\text{Crit}(\tilde{H}) = D$.

Recall now [BPV, p. 142] that such a \tilde{V} can be also blow-down to the trivial fibration $\mathbb{D}(r) \times \mathbb{C}P^1$, in such a way that the singular fiber of \tilde{G} is sent to the regular fiber $\{0\} \times \mathbb{C}P^1$. In other words, that singular fiber is obtained from a regular fiber by a sequence of monoidal transformations. It is then easy to see that D necessarily contains a (-1) -curve: the reason is that a monoidal transformation at a point belonging to an irreducible component of multiplicity m creates a new irreducible component whose multiplicity will be not less than m . By iterating this principle, we see that D contracts to a regular point, whence the first part of the lemma.

Moreover, after this contraction the singular fiber becomes a curve (the fiber G_0 in the now smooth surface V) still dominating a regular fiber, hence in particular it has only normal crossings. Since all the components of G_0 pass through q , we get $k = 1$ (G_0 is a single smooth rational curve of selfintersection 0) or $k = 2$ (G_0 is a pair of two smooth rational curves of selfintersection -1). In the first case q is a regular point, and in the second case it is a Morse type critical point. \square

Remark 3.2. If $H : \mathbb{C}^2 \rightarrow \mathbb{C}$ has an isolated critical point whose Milnor fiber has genus zero, then the critical point is of Morse type: it is a particular case of the previous lemma, but it is also a consequence of classical formulae estimating the genus of the Milnor fiber. However, some care is needed when \mathbb{C}^2 is replaced by a singular surface. For instance, take the function zw on \mathbb{C}^2 and quotient by the involution $(z, w) \mapsto (-z, -w)$. We get a normal surface U and a holomorphic function H on U with an isolated critical point whose Milnor fiber has genus zero. This kind of examples (and more complicated ones) do not appear in Lemma 3.1 because, when we take the resolution $\tilde{U} \rightarrow U$, the critical set of \tilde{H} is not the *full* exceptional divisor D .

We can now return to our compact complex surface S .

Proposition 3.3. *The zero set $Z(\omega)$ is composed only by isolated points, all of Morse type. In particular, the normal bundle $N_{\mathcal{F}}$ coincides with the flat line bundle L .*

Proof. Let D be a connected component of $Z(\omega)$. If it is a curve, then it is a tree of rational curves with negative definite intersection form: this follows from results of Nakamura on the possible configurations of curves on VII₀ surfaces [Nak], and the absence of elliptic curves and cycles of rational curves [Tom] [C-T]. In particular, D is simply connected, and so the (twisted) closed 1-form ω is exact on a neighbourhood \tilde{U} of D : $\omega = d\tilde{H}$ and $\text{Crit}(\tilde{H}) = D$. We therefore are in the setting of Lemma 3.1, and we have just to verify the genus zero hypothesis.

Now, D is contained in a singular fiber M_{ϑ_0} , which can be approximated by regular ones, on which we already know that the foliation has leaves \mathbb{C} or \mathbb{C}^* . It follows obviously that the Milnor fiber has genus zero, and so by Lemma 3.1 the contraction of D produces a smooth point. But we are also assuming since the beginning that S is minimal, hence such a contraction

cannot exist and so $Z(\omega)$ is composed only by isolated points. By a similar argument, and again Lemma 3.1, all these points are of Morse type. \square

4. THE PLANAR CASE

Let $\text{Cv}(f)$ denote the critical values of $f : S \rightarrow \mathbb{S}^1$, and let J be a connected component of $\mathbb{S}^1 \setminus \text{Cv}(f)$. On $f^{-1}(J)$, the foliation \mathcal{F} is nonsingular, and it is given by the transverse intersection of the Kernels of the closed 1-form $df = dF/F$ and the integrable 1-form $d^c F/F$. It follows that the differentiable type of $\mathcal{F}|_{M_\vartheta}$ does not depend on $\vartheta \in J$. We shall say that J is of type \mathbb{C} (resp. type \mathbb{C}^*) if the leaves of \mathcal{F} on $f^{-1}(J)$ are isomorphic to \mathbb{C} (resp. \mathbb{C}^*), according with Proposition 2.1. In this section we shall prove that the existence of a component J of type \mathbb{C} implies that S is a Inoue surface of type S_M .

Let us firstly recast Proposition 3.3 in the context of uniformisation of foliations [Br2]. Recall that the leaves of a singular foliation are *not* simply the leaves outside the singular points: generally speaking, and in order to get a workable definition, we need to compactify some separatrices, the so-called vanishing ends [Br2, p. 732]. However, the presence of a vanishing end implies the existence of a rational curve on the surface, invariant by the foliation, and over which the tangent bundle of the foliation has strictly positive degree [Br2, p. 733]. We claim that this cannot happen in our context, and so the leaves of our \mathcal{F} are just equal to the leaves of $\mathcal{F}|_{S \setminus \text{Sing}(\mathcal{F})}$.

Indeed, if $C \subset S$ is a rational curve, invariant by \mathcal{F} , then $c_1(T_{\mathcal{F}}) \cdot C = 2 - Z(\mathcal{F}, C)$, where $Z(\mathcal{F}, C)$ is the number of the singularities of \mathcal{F} along C [Br1]. On the other hand, since these singularities are all of Morse type (CS residue equal to -1), we also have $C^2 = -Z(\mathcal{F}, C)$. Hence

$$c_1(T_{\mathcal{F}}) \cdot C = 2 + C^2 \leq 0$$

since, by minimality, $C^2 \leq -2$.

We shall also use the main result of [Br2], which says that, since S is not a Hopf surface nor a Kato surface, the foliation \mathcal{F} is *uniformisable*, i.e. does not have vanishing cycles (the reader is invited to give a simple proof of this result, in our very special context).

Proposition 4.1. *Let $J \subset \mathbb{S}^1 \setminus \text{Cv}(f)$ be a component of type \mathbb{C} . Then $J = \mathbb{S}^1$, and S is a Inoue surface of type S_M .*

Proof. Suppose, by contradiction, that $J \neq \mathbb{S}^1$, and let M_{ϑ_0} be a fiber in the boundary of $f^{-1}(J)$. Such a fiber must contain a singular point $p \in \text{Sing}(\mathcal{F})$. Let \mathcal{L} be the leaf corresponding to one of the two separatrices of p , and let $\gamma \subset \mathcal{L}$ be a cycle close to p and turning around it. This cycle, which has no holonomy, can be slightly deformed to a cycle $\gamma' \subset \mathcal{L}'$, where \mathcal{L}' is a leaf contained in M_ϑ , $\vartheta \in J$ close to ϑ_0 . But \mathcal{L}' is simply connected, hence γ' is homotopic to zero in \mathcal{L}' and so γ is homotopic to zero in \mathcal{L} , by the absence of vanishing cycles.

It follows that \mathcal{L} is isomorphic to \mathbb{C} , and $\mathcal{L} \cup \{p\}$ is a rational curve C which contains a unique singularity of the foliation. By the formulae above, $C^2 = -1$, contradicting the minimality of S .

Therefore $J = \mathbb{S}^1$, and so S is a \mathbb{T}^3 -bundle over the circle, without singularities. By [Ino], it is a Inoue surface of type S_M . \square

Remark 4.2. It is worth observing that, in our quite special context, the proof of Inoue's theorem can be highly simplified. Indeed, and by our previous results, on the universal covering \widehat{S} of S the foliation is given by a submersion $\pi : \widehat{S} \rightarrow \mathbb{H}$ (with $\text{Im}(\pi)$ coming from the pluriharmonic function F on \widetilde{S}) all of whose fibers are isomorphic to \mathbb{C} . The key point is to prove that such a universal covering is a product:

$$\widehat{S} = \mathbb{H} \times \mathbb{C}.$$

Indeed, once we know this fact, it remains to study the action of $\Gamma = \pi_1(S)$ on $\mathbb{H} \times \mathbb{C}$. But we already know a lot of properties of such an action (for instance, Γ is a semidirect product of \mathbb{Z} and \mathbb{Z}^3 , which acts on the \mathbb{H} -factor in a special affine way, etc.), and using that knowledge it is easy to conclude that S is a Inoue surface of type S_M .

In order to prove that \widehat{S} is a product, it is sufficient to show that π is a locally trivial fibration, i.e. that every $z \in \mathbb{H}$ has a neighbourhood U_z such that $\pi^{-1}(U_z) = U_z \times \mathbb{C}$. By a classical theorem of Nishino [Nis], this is equivalent to show that $V_z = \pi^{-1}(U_z)$ is Stein. By an argument of Ohsawa [Ohs], the Steinness of V_z follows from the existence of a smooth (not holomorphic!) foliation \mathcal{H} on V_z whose leaves are holomorphic sections of π over U_z (i.e., V_z is trivialisable by a smooth foliation with holomorphic leaves).

Now, in our case such a foliation \mathcal{H} is easy to construct. On any fiber M_ϑ we can take a real analytic foliation by real curves transverse to $\mathcal{F}|_{M_\vartheta}$. By complexifying, we get, on a neighbourhood of M_ϑ , a real analytic foliation by complex curves, transverse to \mathcal{F} . Using the special form of \mathcal{F} , it is easy to see that this foliation, lifted to \widehat{S} , as the required property (here U_z is an horizontal strip in \mathbb{H}).

5. THE CYLINDRICAL CASE

Assume now that every $J \subset \mathbb{S}^1 \setminus \text{Cv}(f)$ is of type \mathbb{C}^* .

Lemma 5.1. *Every leaf of \mathcal{F} is isomorphic to \mathbb{C}^* .*

Proof. The same argument used in Proposition 4.1 shows that, if \mathcal{L} is a leaf in a singular fiber M_{ϑ_0} corresponding to a separatrix of some singular point p , then \mathcal{L} must be diffeomorphic to the cylinder $\mathbb{R} \times \mathbb{S}^1$. Indeed, \mathcal{L} cannot be simply connected by the minimality of S , and cannot have a fundamental group larger than \mathbb{Z} by the absence of vanishing cycles (and of holonomy).

Remark now that, since the foliation is defined by a closed 1-form, every leaf in M_{ϑ_0} is either dense or properly embedded in the complement of the

singularities: this follows from the fact that the holonomy pseudogroup of the foliation is composed only by translations on \mathbb{R} . The second possibility occurs only when the leaf is a “double separatrix”, i.e. a cylinder with both ends converging to singular points, in which case the leaf is isomorphic to \mathbb{C}^* and its closure is a rational curve of selfintersection -2 .

Consider now an arbitrary dense leaf \mathcal{L}' in M_{ϑ_0} . By the absence of vanishing cycles (and of holonomy), we get again that its fundamental group cannot be larger than \mathbb{Z} . On the other hand, this leaf accumulates to the separatrix \mathcal{L} , hence it cannot simply connected otherwise \mathcal{L} would be simply connected too. In conclusion, we see that *every* leaf in M_{ϑ_0} is a cylinder.

To find the conformal type of the leaves, observe that an end of a leaf is either convergent to a singular point (in which case it is obviously parabolic), or it intersects a small ball B around a singular point p along infinitely many annuli. These annuli are not homotopic to zero, by the previous considerations. Moreover, we can extract among them infinitely many ones with bounded modulus (i.e., all isomorphic to $\{r < |z| < 1\}$ with r varying in a compact subset of $(0, 1)$). It follows that the end is parabolic, and the leaf is isomorphic to \mathbb{C}^* . \square

Lemma 5.2. *The line bundle $T_{\mathcal{F}}^{\otimes 2}$ admits a continuous section on $S \setminus \text{Sing}(\mathcal{F})$ which is nowhere vanishing.*

Proof. The complex curve \mathbb{C}^* admits a “almost canonical” holomorphic vector field: the vector field $z \frac{\partial}{\partial z}$, which can be almost uniquely characterized as a complete holomorphic vector field whose flow is $2\pi i$ -periodic. There is however a minor ambiguity, since also the vector field $-z \frac{\partial}{\partial z}$ (conjugate to the previous one by the inversion $z \mapsto 1/z$, which exchanges the two ends) is complete and $2\pi i$ -periodic. This ambiguity can be removed when we take the square: $(z \frac{\partial}{\partial z})^{\otimes 2} = (-z \frac{\partial}{\partial z})^{\otimes 2}$. This means that, given any foliation \mathcal{F} with leaves isomorphic to \mathbb{C}^* , we get a *canonical* nonvanishing section of $T_{\mathcal{F}}^{\otimes 2}$ on $S \setminus \text{Sing}(\mathcal{F})$, by the previous recipe. The point to be proved is that such a section is (at least) continuous.

This is equivalent to prove the following. Let $T \subset S$ be a local transversal to \mathcal{F} , isomorphic to a disc, and let V_T be the corresponding holonomy tube [Br2, p. 734]. It is a complex surface, equipped with a submersion $Q_T : V_T \rightarrow T$, all of whose fibers are isomorphic to \mathbb{C}^* , and a section $q_T : T \rightarrow V_T$. For every $t \in T$ we have a unique isomorphism i_t from $Q_T^{-1}(t)$ to \mathbb{C}^* , sending $q_T(t)$ to 1 (really, there is again a \mathbb{Z}_2 -ambiguity, which however can be easily removed by prescribing an homotopy class). Therefore we get a trivialising map

$$u : V_T \longrightarrow T \times \mathbb{C}^*$$

$$u|_{Q_T^{-1}(t)} = (t, i_t)$$

and the continuity of the above canonical section of $T_{\mathcal{F}}^{\otimes 2}$ is clearly equivalent to the continuity of u (for every transversal T).

As shown in [Ghy, p. 78] (see also [Nis, I.2]), the continuity of u readily follows from Koebe's Theorem. Let us recall the argument, for completeness.

Take a compact $K \subset Q_T^{-1}(t_0)$ and an exhaustion of $Q_T^{-1}(t_0)$ by relatively compact open subsets $\{\Omega_n\}_{n \in \mathbb{N}}$. By a standard argument (e.g. Royden's Lemma), each Ω_n can be holomorphically deformed to the nearby fibers $Q_T^{-1}(t)$, $t \in U_n =$ a neighbourhood of t_0 in T . Thus, the maps i_t , $t \in U_n$, can be seen as all defined on the same domain Ω_n . By Koebe's Theorem, the distortion of i_t on $K \subset \Omega_n$ is uniformly bounded by a constant which tends to zero as $n \rightarrow \infty$, since $Q_T^{-1}(t_0)$ is parabolic. We get in this way that $i_t|_K$ uniformly converge to $i_{t_0}|_K$ as $t \rightarrow t_0$, and since K was arbitrary we get the continuity of u . \square

Remark that, a posteriori, the above map u will be even holomorphic, as well as the canonical section of $T_{\mathcal{F}}^{\otimes 2}$.

It is now easy to complete the proof of Theorem 0.1.

Proposition 5.3. *If every component of $\mathbb{S}^1 \setminus \text{Cv}(f)$ is of type \mathbb{C}^* , then there is only one component, equal to \mathbb{S}^1 , and S is a Inoue surface of type $S_{N,p,q,r;t}^{(+)}$ or $S_{N,p,q,r}^{(-)}$.*

Proof. By Lemma 5.2, the line bundle $T_{\mathcal{F}}^{\otimes 2}$ is topologically trivial, i.e. it is flat. From Proposition 3.3 and $K_S^{-1} = N_{\mathcal{F}} \otimes T_{\mathcal{F}}$ it follows that K_S is flat too, and so $c_1^2(S) = 0$. As explained at the beginning, this is the same as $c_2(S) = 0$, the foliation is nonsingular, and S is a Inoue surface (of the claimed type). \square

As in the planar case, also in the cylindrical case we do not need the full strength of Inoue's theorem, since we can directly prove that a covering of S is isomorphic to $\mathbb{H} \times \mathbb{C}^*$.

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